COHOMOLOGY OF VIRTUALLY NILPOTENT GROUPS WITH COEFFICIENTS IN $\mathbb{R}^k$

BY

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ABSTRACT

We present an explicit description of the cohomology spaces of any finitely generated virtually nilpotent group with (non-trivial) coefficients in a finite-dimensional real vector space. The input of the algorithm we develop to compute these cohomology spaces consists on the one hand of the module structure, and on the other hand of a polynomial crystallographic action of the group. Since any virtually nilpotent group admits such an action (which can be constructed algorithmically) our methods apply to all finitely generated virtually nilpotent groups. As an application of our results, we present explicit formulas for the dimension of the cohomology spaces of a virtually abelian group with coefficients in a finite-dimensional real vector space, equipped with a particular kind of module structure.

1. Introduction

In [7] we used the correspondence between the cohomology of a group G and the de Rham cohomology of a $K(G,1)$-manifold to describe a finite-dimensional cochain complex whose cohomology was exactly the group cohomology of G with trivial real coefficients, for any finitely generated virtually nilpotent group G. This result relied on two classical correspondences: first of all, the de Rham correspondence (see [8]) between the cohomology of differential forms on a $K(G,1)$-manifold and the singular cohomology of this space; secondly, the Eilenberg–Mac Lane correspondence (see, for instance, [2]) between the singular cohomology

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of a $K(G,1)$-space and the group cohomology of G with trivial coefficients in an abelian group. A celebrated theorem of Nomizu (see [12]) ensured that a finite-dimensional subcomplex sufficed to describe this cohomology ring.

In both of the above correspondences, the coefficient module needs to have a trivial G-module structure. When we now turn our attention to cohomology with coefficients in a finite-dimensional real vector space, we need to use a generalisation of these correspondences to cohomology with local coefficients and to cohomology of vector bundle-valued differential forms. Unfortunately, the Nomizu theorem does not have such a generalisation, so we need to tackle this problem differently. Nevertheless, it is interesting to study the connection between cohomology of vector bundle-valued differential forms, singular cohomology with local coefficients and equivariant singular cohomology to have an idea of the type of object that would build up a cochain complex computing the cohomology of a group with coefficients in a finite-dimensional real vector space, equipped with any module structure.

Group cohomology as cohomology of vector bundle-valued differential forms.

Let G be a group acting on a contractible, connected and locally arcwise connected topological space. In case the action of G is free and properly discontinuous, the singular homology chain complex $C_*(X)$ of X (see [11]) turns out to be a free resolution of $\mathbb{Z}$ as a trivial G-module. Therefore, the G-equivariant cohomology of X with coefficients in a G-module M is nothing but the group cohomology of G with coefficients in M,

$$E^*_G(X,M) = H^*(G,M). \tag{1}$$

For a trivial module M, the Eilenberg–Mac Lane correspondence states that the G-equivariant cohomology $E^*_G(X,M)$ of X with coefficients in M is isomorphic to the singular cohomology of the quotient space X/G with coefficients in M. To incorporate a non-trivial module structure, we need to replace this singular cohomology by cohomology with local coefficients. According to [14, Chapter VI, Theorem 3.4*], the above correspondence carries over to cohomology with local coefficients. The G-equivariant cohomology of X with coefficients in a G-module M turns out to be isomorphic to the cohomology of X/G with respect to the system of local coefficients $M(\cdot)$ arising from the G-action on M,

$$E^*_G(X,M) \cong H^*(X/G, M(\cdot)). \tag{2}$$

Now let X be a differentiable manifold, and suppose G acts on X by diffeomorphisms. Then the quotient manifold X/G inherits a differentiable structure, and the de Rham cohomology of differential forms on X/G proves to be isomorphic to the singular cohomology of X/G with trivial real coefficients. When considering finite-dimensional real vector spaces $\mathbb{R}^k$ carrying a non-trivial G-module structure, we are led to the cohomology of vector bundle-valued forms on X/G. The G-module structure of $\mathbb{R}^k$ gives rise to a vector bundle E with a flat connection, thus determining a cochain complex of forms with values in this vector bundle. Since the classical de Rham theorem for a (para)compact quotient manifold X/G, describing the isomorphism between cohomology of differential forms on X/G and singular cohomology of X/G with trivial coefficients in $\mathbb{R}$, generalises to this broader setting, we obtain an isomorphism between the cohomology of differential forms on X/G with values in E and the cohomology with respect to the corresponding system $E'(\cdot)$ of local coefficients,

$$H^*_{deRham}(X/G, E) \cong H^*_{local}(X/G, E'(\cdot)). \tag{3}$$

Putting together the isomorphisms (1), (2) and (3), we conclude that the group cohomology of G with coefficients in a finite-dimensional real vector space $\mathbb{R}^k$ carrying any module structure matches the cohomology of the complex of differential forms with values in the vector bundle E corresponding to the module structure of $\mathbb{R}^k$,

$$H^*(G, \mathbb{R}^k) \cong H^*_{deRham}(X/G, E). \tag{4}$$

In the remainder of this introduction, we describe the complex of differential forms on X/G with values in the vector bundle associated to the G-module structure of $\mathbb{R}^n$ algebraically and in an equivariant setting. This allows us to define, in analogy with the trivial real coefficient case, a finite-dimensional subcomplex whose cohomology turns out to be exactly the group cohomology of G with coefficients in $\mathbb{R}^k$, in case G is torsion free, finitely generated and nilpotent and the $\mathbb{R}^k$-module structure is unipotent. This result is accomplished in section 2. We then generalise to the case of arbitrary module structures for $\mathbb{R}^k$ in section 3, and to the case of virtually nilpotent groups in section 4. In the last section, we use this description of the cohomology of a virtually nilpotent group to draw up explicit formulas for the dimension of the cohomology spaces of a virtually abelian group with coefficients in $\mathbb{R}^k$ equipped with a particular type of module structure.

An algebraic description of the complex of vector-bundle valued differential forms.

Let G be a group acting cocompactly, freely and properly discontinuously on a contractible, connected and locally arcwise connected differentiable manifold

X via $\rho\colon G \to \mathcal{D}(X)$, where $\mathcal{D}(X)$ is the group of all diffeomorphisms of X. Suppose G acts on $\mathbb{R}^k$ via $\varphi\colon G \to \mathrm{GL}(k,\mathbb{R})$. Let

$$\Omega^*(X,\mathbb{R}^k)$$

be the vector space of all k-tuples of differential forms on X. A k-tuple ${}^t(\omega_1,\ldots,\omega_k) \in \Omega^*(X,\mathbb{R}^k)$ of differential forms on X is called $(\rho(G),\varphi(G))$-invariant if

$$\begin{pmatrix} \rho(g^{-1})^*\omega_1 \\ \vdots \\ \rho(g^{-1})^*\omega_k \end{pmatrix} = \varphi(g) \begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \end{pmatrix},$$

where $\rho(g)^*$ denotes pullback under the action of g, and the right-hand side is given by formal matrix multiplication. The vector space of $(\rho(G),\varphi(G))$-invariant k-tuples of differential p-forms we denote by

$$\Omega^p(X,\mathbb{R}^k)^{\rho(G),\varphi(G)}.$$

The layerwise differentiation

$$d\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \end{pmatrix} = \begin{pmatrix} d\omega_1 \\ \vdots \\ d\omega_k \end{pmatrix}$$

turns the graded vector space $\Omega^*(X,\mathbb{R}^k)^{\rho(G),\varphi(G)}$ into a cochain complex

$$0 \longrightarrow \Omega^0(X,\mathbb{R}^k)^{\rho(G),\varphi(G)} \xrightarrow{\ d\ } \Omega^1(X,\mathbb{R}^k)^{\rho(G),\varphi(G)} \xrightarrow{\ d\ } \cdots.$$

When computing the group cohomology of G using the singular homology resolution $C_*(X)$ of X as a free resolution, the isomorphism in (4) is induced by the k-fold integral $\mathcal{I}$ of a k-tuple of forms

$$(5)\qquad \mathcal{I}\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \end{pmatrix} : C_*(X) \to \mathbb{R}^k \quad \text{defined by} \quad \mathcal{I}\begin{pmatrix} \omega_1 \\ \vdots \\ \omega_k \end{pmatrix}(T) = \begin{pmatrix} \int_T \omega_1 \\ \vdots \\ \int_T \omega_k \end{pmatrix}$$

over non-degenerate singular cubes $T\colon [0,1]^* \to X$ on X.

An interesting subcomplex of $\Omega^*(\mathbb{R}^n,\mathbb{R}^k)^{\rho(G),\varphi(G)}$.

Inspired by [7] we specify the manifold X and the type of action ρ, and thus restrict to groups allowing a so-called polynomial crystallographic action, and, in particular, to finitely generated virtually nilpotent groups.

Let $n > 0$. A polynomial diffeomorphism $p: \mathbb{R}^n \to \mathbb{R}^n$ of $\mathbb{R}^n$ is a bijective map such that both p and its inverse p^{-1} can be expressed as polynomials. The group of polynomial diffeomorphisms is denoted by $\mathcal{P}(\mathbb{R}^n)$. An action $\rho: G \to \mathcal{P}(\mathbb{R}^n)$ of a group G on $\mathbb{R}^n$ via polynomial diffeomorphisms is called **polynomial crystallographic** if it is both properly discontinuous and cocompact. Moreover, the action is **of bounded degree** if there exists an integer M such that the degrees of all maps in $\rho(G)$ are bounded above by M.

In [4] (see also [3]) it is shown that any polycyclic-by-finite group, and hence any finitely generated virtually nilpotent group, admits a polynomial crystallographic action of bounded degree. Moreover, this action is unique up to conjugation inside $\mathcal{P}(\mathbb{R}^n)$ (see [1]) and can be constructed algorithmically, for instance, starting from a presentation of the group (see [6], [5] or again [4]).

Let $\rho: G \to \mathcal{P}(\mathbb{R}^n)$ be a polynomial crystallographic action of G on $\mathbb{R}^n$, and $\varphi: G \to \mathrm{GL}(k, \mathbb{R})$ a G-module structure for $\mathbb{R}^k$. In analogy with the main result of [7], we define the subcomplex

$$\Omega^*_P(X, \mathbb{R}^k)^{\rho(G),\varphi(G)}$$

of the complex $\Omega^*(X, \mathbb{R}^k)^{\rho(G),\varphi(G)}$ as the space of all of $(\rho(G), \varphi(G))$-invariant k-tuples of forms on $\mathbb{R}^n$ having *polynomial* coordinate functions. Using the fact that, for a polycyclic-by-finite group, any two polynomial crystallographic actions of bounded degree are polynomially conjugated (see [1]), it is easy to show that this space does not depend on the choice of polynomial structure.

For any finitely generated virtually nilpotent group G, we relate the cohomology of this restricted complex of tuples of differential forms to the cohomology of G with unipotent coefficients in $\mathbb{R}^k$, and, in doing so, gain some understanding about what this complex $\Omega^*_P(X, \mathbb{R}^k)^{\rho(G),\varphi(G)}$ looks like.

2. Cohomology of $\mathcal{T}$-groups with unipotent coefficients

Let N be a $\mathcal{T}$-group, that is, a finitely generated torsion-free nilpotent group. Then lemma 1.1 in [9] implies that unipotent coefficient modules with a vector space structure have a particular behaviour compared to general coefficient modules when considering cohomology (see section 3 for a more detailed discussion of this observation). Moreover, the fact that a unipotent N-module structure of $\mathbb{R}^k$ given by $\varphi: N \to \mathrm{GL}(k, \mathbb{R})$ is upper triangular up to a change of basis, makes this setting extremely well fit for an inductive argument. With these ideas in mind we prove

THEOREM 2.1: *Let $\rho\colon N \to \mathcal{P}(\mathbb{R}^n)$ be a polynomial crystallographic action of a $\mathcal{T}$-group N on $\mathbb{R}^n$, and $\varphi\colon N \to \mathrm{GL}(k,\mathbb{R})$ a unipotent N-module structure for $\mathbb{R}^k$. Then the cochain map*

$$\mathcal{I}\colon \Omega^*_P(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)} \to \mathrm{Hom}_{\mathbb{Z}G}(C_*(\mathbb{R}^n),\mathbb{R}^k)$$

induces an isomorphism on cohomology,

$$H^*(\Omega^*_P(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)}) \cong H^*_\varphi(N,\mathbb{R}^k).$$

To establish this isomorphism, we first of all describe the inductive setup, consisting of the obvious decomposition of $\mathbb{R}^k$ as an N-module, and the equivalent result for the complex of tuples of forms, which requires some understanding of the way this complex is built up. We then use these ingredients to work out a proof of the theorem in section 2.2.

2.1 A SHORT EXACT SEQUENCE OF COMPLEXES OF FORMS. Since the module structure φ of $\mathbb{R}^k$ is unipotent, we may safely assume $\varphi(N)$ consists of upper triangular matrices, that is, $\varphi(N) \subseteq \mathrm{Tr}_1(k,\mathbb{R})$. Then the first component of $\mathbb{R}^k$ is an N-submodule, say I, upon which N acts trivially. The induced module structure $\bar\varphi\colon N \to \mathrm{Tr}_1(k-1,\mathbb{R})$ of the quotient space $V = \mathbb{R}^k/I$ is defined by

$$\varphi(n) = \left(\begin{array}{c|ccc} 1 & * & \dots & * \\ \hline 0 & & & \\ \vdots & & \bar\varphi(n) & \\ 0 & & & \end{array}\right)$$

for all $n \in N$. We end up with a short exact sequence of N-modules,

$$0 \longrightarrow I \longrightarrow \mathbb{R}^k \longrightarrow V \longrightarrow 0. \tag{6}$$

We now describe the the corresponding result for invariant tuples of polynomial forms. For any $0 \le p \le n$, let

$$i\colon \Omega^p_P(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}} \to \Omega^p_P(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)}\colon \omega \to \begin{pmatrix} \omega \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

be the inclusion of the first layer, and

$$\mathrm{proj}: \Omega^p_P(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)} \to \Omega^p_P(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar\varphi(N)}\colon \begin{pmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_k \end{pmatrix} \mapsto \begin{pmatrix} \omega_2 \\ \vdots \\ \omega_k \end{pmatrix}$$

the projection on the last $k-1$ components. Note that $\Omega_P^*(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}}$ is nothing but the space of $\rho(N)$-invariant polynomial forms on $\mathbb{R}^n$. Clearly, i is injective and $\mathrm{Im}(i)=\mathrm{Ker}(\mathrm{proj})$. But there is more:

THEOREM 2.2: *The sequence*

$$0 \longrightarrow \Omega_P^*(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}} \xrightarrow{\ i\ } \Omega_P^*(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)} \xrightarrow{\ \mathrm{proj}\ } \Omega_P^*(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)} \longrightarrow 0$$

is exact.

The surjectivity of proj is the only thing left to prove. This is readily checked by showing that

$$\dim\Omega_P^*(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)}=\dim\Omega_P^*(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}}+\dim\Omega_P^*(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)},$$

which is an immediate consequence of

PROPOSITION 2.3: *Let N be a $\mathcal{T}$-group and $\rho\colon N\to P(\mathbb{R}^n)$ a polynomial crystallographic action of N on $\mathbb{R}^n$. Suppose $\varphi\colon N\to \mathrm{Tr}_1(k,\mathbb{R})$ gives $\mathbb{R}^k$ a unipotent N-module structure. Then*

$$\dim\Omega_P^*(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)}=k\binom{n}{p}$$

for any $0\le p\le n$.

Proof: We proceed by induction on k. The case $k=1$ was treated in [7, Theorem 3.5], so let $k>1$ and assume the proposition is valid for all coefficient modules up to dimension $k-1$.

Let G be the Mal'cev completion of N, that is, the unique simply connected and connected nilpotent Lie group containing N as a uniform lattice. As $\mathrm{Tr}_1(k,\mathbb{R})$ is a simply connected nilpotent Lie group, there exists a unique extension $\tilde{\varphi}\colon G\to \mathrm{Tr}_1(k,\mathbb{R})$ of $\varphi\colon N\to \mathrm{Tr}_1(k,\mathbb{R})$. According to theorem 3.2 in [7], the polynomial crystallographic action $\rho\colon N\to\mathcal{P}(\mathbb{R}^n)$ extends also in a unique way to a simply transitive action $\tilde{\rho}\colon G\to\mathcal{P}(\mathbb{R}^n)$ of G on $\mathbb{R}^n$, and the evaluation map

$$\mathrm{Ev}\colon G\to\mathbb{R}^n\colon g\mapsto\tilde{\rho}(g)(0)$$

turns out to be a diffeomorphism. As G is nilpotent, the composition $\mathrm{Ev}\circ\exp\colon\mathcal{G}\to\mathbb{R}^n$ is polynomial in the coordinates of the Lie algebra $\mathcal{G}$ of G. Moreover, lemma 2 from [1] shows that its inverse is also polynomial.

It is now easy to see that the mapping

$$F\colon \mathbb{R}^n \to \mathrm{Tr}_1(k,\mathbb{R})\colon x \to \tilde{\varphi}(\mathrm{Ev}^{-1}(x))$$

is polynomial, as G and $\mathrm{Tr}_1(k,\mathbb{R})$ are nilpotent. Moreover, the mapping

$$D\colon \mathbb{R}^n \to P(\mathbb{R}^n)\colon x \to \tilde{\rho}(\mathrm{Ev}^{-1}(x))$$

is also polynomial, in the sense that the coefficients of the polynomials $\tilde{\rho}(\mathrm{Ev}^{-1}(x))$ depend polynomially on x.

The functions D and F allow us to build a subspace of $(\rho(N),\varphi(N))$-invariant polynomial p-forms. Let $p \in \mathbb{N}$ and $c_1,\dots,c_k \in \mathrm{Alt}^p(\mathbb{R}^n)$ be alternating p-forms on $\mathbb{R}^n$. Then define

$$\omega_{c_1,\dots,c_k}(x)(\xi^{(1)},\dots,\xi^{(p)}) = F(x)\cdot \begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix} (D(x)_*^{-1}\xi^{(1)},\dots,D(x)_*^{-1}\xi^{(p)}). \tag{7}$$

Evidently, $\omega_{c_1,\dots,c_k}$ is polynomial because the functions F and D are. Moreover, let $n \in N$ and $g \in G$ such that $x = \mathrm{Ev}(g)$. Then

$$\begin{aligned}
\rho(n)^*(\omega_{c_1,\dots,c_k})(x)(\xi^{(1)},\dots,\xi^{(p)}) &= \omega_{c_1,\dots,c_k}(\rho(ng)(0))(\rho(n)_*\xi^{(1)},\dots,\rho(n)_*\xi^{(p)})\\
&= \tilde{\varphi}(ng)\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}(\tilde{\rho}(ng)_*^{-1}\rho(n)_*\xi^{(1)},\dots,\tilde{\rho}(ng)_*^{-1}\rho(n)_*\xi^{(p)})\\
&= \varphi(n)F(x)\begin{pmatrix} c_1 \\ \vdots \\ c_k \end{pmatrix}(D(x)_*^{-1}\xi^{(1)},\dots,D(x)_*^{-1}\xi^{(p)})\\
&= \varphi(n)\omega_{c_1,\dots,c_k}(x)(\xi^{(1)},\dots,\xi^{(p)})
\end{aligned}$$

for any $x \in \mathbb{R}^n$ and any vectors $\xi^{(1)},\dots,\xi^{(p)}$ tangent to $\mathbb{R}^n$ at x. Therefore, $\omega_{c_1,\dots,c_k}$ is $(\rho(N),\varphi(N))$-invariant. Clearly, all k-tuples of p-forms defined as above form a vector space which is isomorphic to $[\mathrm{Alt}^p(\mathbb{R}^n)]^k$, so its dimension is $k\binom{n}{p}$. Summarizing, $\Omega_P^*(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)}$ has dimension at least $k\binom{n}{p}$.

Since

$$\dim \Omega_P^*(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)} = \dim \mathrm{Ker}(\mathrm{proj}) + \dim \mathrm{Im}(\mathrm{proj}),$$

where $\mathrm{Im}(\mathrm{proj}) \subseteq \Omega_P^p(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)}$ and

$$\dim \mathrm{Ker}(\mathrm{proj}) = \dim \Omega_P^p(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}},$$

the induction hypothesis implies that

$$\dim \Omega^*_P(\mathbb{R}^n, \mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)} \leq \binom{n}{p} + (k-1)\binom{n}{p} = k\binom{n}{p}.$$

Together with the previous step, this proves the proposition. ∎

Note that the first part of the above proof gives us an explicit formula describing the space $\Omega^p_P(\mathbb{R}^n, \mathbb{R}^k)^{\rho(N),\varphi(N)}$, for any $p \geq 0$.

2.2 A PROOF OF THEOREM 2.1. Once we have the short exact sequence (6) and the one from Theorem 2.2, the proof of Theorem 2.1 consists of a straightforward induction argument on the dimension of the coefficient space $\mathbb{R}^k$ using standard techniques.

Proof of Theorem 2.1: If $k = 1$ then $\varphi\colon N \to \mathrm{Tr}_1(1,\mathbb{R})$ is just the trivial representation, so in this case theorem 3.8 from [7] shows that

$$H^*_{triv}(N,\mathbb{R}) \cong H^*(\Omega^*_P(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}}).$$

Now suppose the theorem has been proven for coefficient modules of dimension smaller than or equal to $k-1$.

The short exact sequence (6) of coefficient modules induces a long exact sequence in cohomology (see, for instance, [2, prop. 6.1])

$$\cdots \longrightarrow H^{p-1}(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n), V)) \longrightarrow H^p(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n), I))$$

$$\longrightarrow H^p(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n), \mathbb{R}^k)) \longrightarrow H^p(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n), V))$$

$$\longrightarrow H^{p+1}(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n), I)) \longrightarrow \cdots.$$

Analogously, the short exact sequence from Theorem 2.2 yields a long exact sequence in cohomology (see, for instance, [10, Ch. II, theorem 4.1])

$$\cdots \longrightarrow H^{p-1}(\Omega^*_P(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)}) \longrightarrow H^p(\Omega^*_P(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}})$$

$$\longrightarrow H^p(\Omega^*_P(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)}) \longrightarrow H^p(\Omega^*_P(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)})$$

$$\longrightarrow H^{p+1}(\Omega^*_P(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}}) \longrightarrow \cdots.$$

The cochain map

$$\mathcal{I}\colon \Omega^*_P(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)} \to \mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n,\mathbb{R}^k)$$

defined by (5) now induces the vertical mappings in the following commutative diagram:

$$\begin{array}{ccccc}
H^{p-1}(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n),V)) & \longrightarrow & H^p(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n),I)) & \longrightarrow & H^p(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n),\mathbb{R}^k)) \\
\uparrow \mathcal{I}_V^{p-1} & & \uparrow \mathcal{I}_I^p & & \uparrow \mathcal{I}^p \\
H^{p-1}(\Omega_P^*(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)}) & \longrightarrow & H^p(\Omega_P^*(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}}) & \longrightarrow & H^p(\Omega_P^*(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)})
\end{array}$$

$$\begin{array}{ccccc}
\longrightarrow & H^p(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n),V)) & \longrightarrow & H^{p+1}(\mathrm{Hom}_{\mathbb{Z}N}(C_*(\mathbb{R}^n),I)) \\
& \uparrow \mathcal{I}_V^p & & \uparrow \mathcal{I}_I^{p+1} \\
\longrightarrow & H^p(\Omega_P^*(\mathbb{R}^n,\mathbb{R}^{k-1})^{\rho(N),\bar{\varphi}(N)}) & \longrightarrow & H^{p+1}(\Omega_P^*(\mathbb{R}^n,\mathbb{R})^{\rho(N),\mathrm{triv}})
\end{array}$$

where each of the morphims $\mathcal{I}$, $\mathcal{I}_I$ and $\mathcal{I}_V$ are defined as in (5), each one with respect to the appropriate N-module.

Because of theorem 3.8 in [7], $\mathcal{I}_I^p$ is an isomorphism, and the induction hypothesis ensures that $\mathcal{I}_V^p$ is an isomorphism as well. Applying the Five Lemma [10, Ch. 1, lemma 3.3] now shows that the middle map $\mathcal{I}^p$ is also an isomorphism. We repeat this for every $p \in \{0, \ldots, n\}$ to find the desired isomorphism of cohomology spaces. ∎

We illustrate our construction by means of a simple example, disregarding the fact that cohomology could also be computed differently in this easy case. A more serious example is given in Example 3.1.

Example 2.1: Consider the group $\mathbb{Z}^2 = \langle e_1, e_2 \rangle$ with its translation action $\rho\colon \mathbb{Z}^2 \to \mathcal{P}(\mathbb{R}^2)$ given by

$$\rho(z_1e_1 + z_2e_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + z_1 \\ x_2 + z_2 \end{pmatrix}$$

as a polynomial crystallographic action of $\mathbb{Z}^2$ on $\mathbb{R}^2$. Suppose the $\mathbb{Z}^2$-module structure of $\mathbb{R}^3$ is given by

$$\varphi(e_1) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the extension $\tilde{\rho}$ of ρ to the Mal'cev completion $\mathbb{R}^2$ of $\mathbb{Z}^2$ is just

$$\tilde{\rho}(y_1, y_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

for any $(y_1, y_2) \in \mathbb{R}^2$. The extension $\tilde{\varphi}$ of φ to $\mathbb{R}^2$ is defined by

$$\tilde{\varphi}(y_1, y_2) = \begin{pmatrix} 1 & 0 & y_1 \\ 0 & 1 & y_2 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now let us compute the differential forms using formula (7).

	General form	Differential
0-forms	$\begin{pmatrix} c_1 + x_1 c_3 \\ c_2 + x_2 c_3 \\ c_3 \end{pmatrix}$	$\begin{pmatrix} c_3 dx_1 \\ c_3 dx_2 \\ 0 \end{pmatrix}$
1-forms	$\begin{pmatrix} c_{1,1} dx_1 + c_{1,2} dx_2 + x_1(c_{3,1} dx_1 + c_{3,2} dx_2) \\ c_{2,1} dx_1 + c_{2,2} dx_2 + x_2(c_{3,1} dx_1 + c_{3,2} dx_2) \\ c_{3,1} dx_1 + c_{3,2} dx_2 \end{pmatrix}$	$\begin{pmatrix} c_{3,2} dx_1 \wedge dx_2 \\ -c_{3,1} dx_1 \wedge dx_2 \\ 0 \end{pmatrix}$
2-forms	$\begin{pmatrix} c_1 dx_1 \wedge dx_2 + x_1 c_3 dx_1 \wedge dx_2 \\ c_2 dx_1 \wedge dx_2 + x_2 c_3 dx_1 \wedge dx_2 \\ c_3 dx_1 \wedge dx_2 \end{pmatrix}$	0

For the cohomology spaces we find

$$H^0_\varphi(\mathbb{Z}^2, \mathbb{R}^3) \cong \mathbb{R}^2, \quad H^1_\varphi(\mathbb{Z}^2, \mathbb{R}^3) \cong \mathbb{R}^3, \quad H^2_\varphi(\mathbb{Z}^2, \mathbb{R}^3) \cong \mathbb{R}.$$

3. Cohomology of $\mathcal{T}$-groups with general coefficients

As suggested before, the unipotent module structures are in a sense the essential ones when considering cohomology of $\mathcal{T}$-groups. In this section we show that the cohomology of a $\mathcal{T}$-group with coefficients in a vector space with any module structure reduces to the cohomology of this group with coefficients in the maximal unipotent part of this module. Using this reduction, the cohomology description of the preceding section also applies to cohomology with non-unipotent coefficients.

Let N be any group. First of all, we show that each finite-dimensional N-module contains a maximal unipotent submodule. Although this proposition is well-known and quite easy, we present it here with a complete proof, to illustrate the fact that the determination of this maximal unipotent submodule uses an easy algorithm. A fairly standard argument then shows how, for any $\mathcal{T}$-group, cohomology with coefficients in $\mathbb{R}^k$ reduces to cohomology with coefficients in the maximal unipotent submodule of $\mathbb{R}^k$. As an example, we compute the cohomology of the free 2-step nilpotent group on 4 generators with coefficients in a 4-dimensional real vector space.

PROPOSITION 3.1: *Let N be a group acting on $\mathbb{R}^k$ via $\varphi: N \to \mathrm{GL}(k,\mathbb{R})$. Then $\mathbb{R}^k$ has a N-invariant subspace M such that*

(1) *the action of N on M is unipotent, and*
(2) *the induced action on $\mathbb{R}^k/M$ has no non-trivial fixed points.*

Proof: We prove this proposition by induction on k. If $k = 1$, then either φ is trivial, and $M = \mathbb{R}$, or φ isn't, but then the N-action has no non-trivial fixed points, and $M = 0$. Now suppose we know how to find such a maximal unipotent submodule in any N-module with a real vector space structure of dimension smaller than k, and consider an N-module of dimension k. If the N-action has no non-trivial fixed points, then $M = 0$. Otherwise, let $v \neq 0 \in \mathbb{R}^k$ be a fixed point. Then the subspace $\mathrm{vct}\{v\}$ spanned by v is an N-submodule, and by induction there exists a submodule $\bar{M}$ of the quotient space $\mathbb{R}^k/\mathrm{vct}\{v\}$, equipped with the induced module structure, such that the action of N on $\bar{M}$ is unipotent and the induced action on $\mathbb{R}^{k-1}/\bar{M}$ has no non-trivial fixed points. Now $M = \{w \in \mathbb{R}^k |\, w + \mathrm{vct}\{v\} \in \bar{M}\}$ is a N-submodule with unipotent module structure, and the induced action on $\mathbb{R}^k/M = \mathbb{R}^{k-1}/\bar{M}$ has no non-trivial fixed points. ∎

THEOREM 3.2: *Let N be a $\mathcal{T}$-group acting on $\mathbb{R}^k$ via a morphism $\varphi: N \to \mathrm{GL}(k,\mathbb{R})$. Suppose M is the maximal unipotent submodule of $\mathbb{R}^k$, with an N-action given by $\varphi_U: N \to \mathrm{GL}(M)$. Then*

$$H^*_\varphi(N,\mathbb{R}^k) \cong H^*_{\varphi_U}(N,M).$$

Proof: When taking cohomology with respect to the short exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{R}^k \longrightarrow \mathbb{R}^k/M \longrightarrow 0$$

we obtain the long exact sequence

$$\cdots \to H^{p-1}(N,\mathbb{R}^k/M) \to H^p(N,M) \to H^p(N,\mathbb{R}^k) \to H^p(N,\mathbb{R}^k/M) \to \cdots.$$

But by lemma 1.1 in [9], $H^p(N,\mathbb{R}^k/M) = 0$ for all p, because N acts on $\mathbb{R}^k/M$ without non-trivial fixed points. Therefore,

$$H^p_\varphi(N,\mathbb{R}^k) \cong H^p_{\varphi_U}(N,M)$$

for any $0 \leq p \leq n$. ∎

To describe the cohomology spaces of a $\mathcal{T}$-group with coefficients in a finite-dimensional real vector space equipped with any module structure, we just compute the maximal unipotent submodule and draw up the complex of invariant tuples of polynomial differential forms, as in section 2.

Example 3.1: Let $F_{2,4}$ be the free 4-generated 2-step nilpotent group

$$F_{2,4} = \langle e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10} || [e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_1, e_4] = e_7, [e_2, e_3] = e_8, [e_2, e_4] = e_9, [e_3, e_4] = e_{10}, \rangle.$$

Then $F_{2,4}$ has a polynomial crystallographic action $\rho\colon F_{2,4} \to \mathcal{P}(\mathbb{R}^{10})$ given by

$$\rho(e_1^{u_1} \ldots e_{10}^{u_{10}}) \begin{pmatrix} x_1 \\ \vdots \\ x_{10} \end{pmatrix} = \begin{pmatrix} x_1 + u_1 \\ x_2 + u_2 \\ x_3 + u_3 \\ x_4 + u_4 \\ x_5 + u_5 + x_1 u_2 \\ x_6 + u_6 + x_1 u_3 \\ x_7 + u_7 + x_1 u_4 \\ x_8 + u_8 + x_2 u_3 \\ x_9 + u_9 + x_2 u_4 \\ x_{10} + u_{10} + x_3 u_4 \end{pmatrix}.$$

Define a module structure $\varphi\colon F_{2,4} \to \mathrm{GL}(4, \mathbb{R})$ by choosing

$$\varphi(e_1) = \begin{pmatrix} -1 & 2 & 0 & -2 \\ 2 & -1 & 0 & 2 \\ -2 & 2 & 1 & -2 \\ 4 & -4 & 0 & 5 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} -3 & 0 & 4 & 0 \\ 2 & 1 & -2 & 0 \\ -4 & 0 & 5 & 0 \\ 6 & 0 & -6 & 1 \end{pmatrix},$$

$$\varphi(e_3) = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & -1 \\ 2 & -2 & 0 & 3 \end{pmatrix}, \quad \varphi(e_4) = \begin{pmatrix} 3 & 0 & -2 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & 0 & -1 & 0 \\ -3 & 0 & 3 & 1 \end{pmatrix}.$$

Then the unipotent submodule has dimension 2, and the cohomology spaces of the group $F_{2,4}$ with coefficients in $\mathbb{R}^4$ (or equivalently, in its unipotent submodule) are computed analogously to the methods described in Example 2.1. For their dimensions we find

$$\begin{aligned} &H^0_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^2, && H^1_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{11}, && H^2_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{46}, \\ &H^3_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{122}, && H^4_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{201}, && H^5_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{232}, \\ &H^6_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{201}, && H^7_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{122}, && H^8_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{46}, \\ &H^9_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^{10}, && H^{10}_\varphi(F_{2,4}, \mathbb{R}^4) \cong \mathbb{R}^2. \end{aligned}$$

4. Cohomology of virtually nilpotent groups

In this section we put to work the description we gave of the cohomology of a $\mathcal{T}$-group with coefficients in $\mathbb{R}^k$, and show how these techniques can be extended to compute the cohomology of a finite extension of a $\mathcal{T}$-group. It is well known (see, for instance, [13]) that any finitely generated virtually nilpotent group has a $\mathcal{T}$-subgroup of finite index.

As a first step, we sketch the connection between the cohomology of a group and of a subgroup of finite index, under rather mild conditions on the coefficient module.

Proposition 4.1: *Let E be a group and N a finite index subgroup of E. Suppose A is an E-module uniquely divisible by the index of N in E. Then*

$$H^*(E,A) \cong H^*(N,A)^{E/N},$$

where the E/N-action on $H^*(N,A)$ is induced by conjugation in E.

For a completely worked-out proof of this proposition we refer to [7].

We now use this relation to give a description of the cohomology spaces of a virtually nilpotent group. Let E be a finitely generated virtually nilpotent group, N a $\mathcal{T}$-subgroup of finite index, and suppose $\varphi\colon E \to \mathrm{GL}(k,\mathbb{R})$ gives $\mathbb{R}^k$ an E-module structure. According to the above proposition, we need to understand the action of the finite quotient E/N (or, equivalently, of E itself) on the cohomology of N. Of course, this cohomology we would like to compute as the cohomology of the complex

$$\Omega^*_{\mathcal{P}}(\mathbb{R}^n,\mathbb{R}^k)^{\rho(N),\varphi(N)}$$

of invariant k-tuples of polynomial forms on $\mathbb{R}^n$, where $\rho\colon N \to \mathcal{P}(\mathbb{R}^n)$ is a polynomial crystallographic action of N on $\mathbb{R}^n$. To do this we first of all need to restrict to the maximal unipotent N-submodule M of $\mathbb{R}^k$. The fact that M is also an E-submodule, thus ensuring that the E/N-action on the cohomology of N with coefficients in $\mathbb{R}^k$ carries over to the cohomology of N with coefficients in M, is ensured by

Lemma 4.2 (with notation as above): *The maximal unipotent N-submodule M of $\mathbb{R}^k$ is an E-submodule.*

Proof: The proof of Proposition 3.1 shows the iterative build-up of M,

$$0 \subset M^{(1)} \subset M^{(2)} \subset \cdots \subset M^{(r)} = M,$$

where $M^{(i)}/M^{(i-1)}$ is the space of fixed points under the induced action of N on $\mathbb{R}^k/M^{(i-1)}$. We show M is an E-module by induction on r.

If $r = 1$ then M is just the space of fixpoints under the action of N, so, for any $e \in E$, $v \in M$ and $n \in N$,

$$^{n}(^{e}v) = {}^{e(e^{-1}ne)}v = {}^{e}v$$

as N is normal in E. Therefore ^{e}v is again fixed under the action of N. Now suppose $M^{(r-1)}$ is an E-module. Then $\mathbb{R}^k/M^{(r-1)}$ is, and analogously to the case $r = 1$ we can show that $M^{(r)}/M^{(r-1)}$ is an E-module, thus $M^{(r)}$ is as well. ∎

To translate the E-action on cohomology to an action of E on this particular complex of tuples of forms, we first of all need to extend the polynomial crystallographic action ρ of N on $\mathbb{R}^n$ to an action of E on $\mathbb{R}^n$. According to [7,Prop. 4.2], such an extension $\tilde{\rho}$: $E \to \mathcal{P}(\mathbb{R}^n)$ always exists, and is uniquely determined. We then translate the action induced by conjugation inside E, which is canonically defined on any projective resolution of $\mathbb{Z}$ as a trivial N-module with an E-module structure (see, for instance, [2]), to an action on the complex of $(\rho(N), \varphi(N)$-invariant. The result of this translation, which is conducted along the lines of the translation of this same action in the case $k = 1$ and $\varphi(N) = \{1\}$ in [7], is given in

PROPOSITION 4.3 (with notation as above): *Let*

$$\omega = {}^{t}\,(\,\omega_1 \quad \ldots \quad \omega_k\,) \in \Omega^*_P(\mathbb{R}^n, \mathbb{R}^k)^{\rho(N),\varphi(N)}.$$

The action of an $e \in E$ on ω is given by

$$^{e}\omega = \varphi(e) \cdot \begin{pmatrix} \tilde{\rho}(e^{-1})^*\omega_1 \\ \vdots \\ \tilde{\rho}(e^{-1})^*\omega_k \end{pmatrix}.$$

Computing the cohomology of a virtually nilpotent group E now comes down to computing the invariants in the cohomology of a normal $\mathcal{T}$-subgroup of finite index under the action described in the above proposition.

THEOREM 4.4: *Let E be a finitely generated virtually nilpotent group with a polynomial crystallographic action $\tilde{\rho}$: $E \to \mathcal{P}(\mathbb{R}^n)$, and N a normal $\mathcal{T}$-subgroup of finite index. Let $\mathbb{R}^k$ be an E-module via φ: $E \to GL(k, \mathbb{R})$ and M its maximal unipotent N-submodule with N-action given by φ_U: $N \to Tr_1(k, \mathbb{R})$. Then*

$$H^*_\varphi(E, \mathbb{R}^k) \cong H^*[(\Omega^*_P(\mathbb{R}^n, M)^{\tilde{\rho}(E),\varphi_U(E)})].$$

In this theorem the E/N-action is pulled into the cohomology functor, thus computing invariants before taking cohomology. This is indeed just a concise form for the space of the invariants in the cohomology of N under the action of the finite group E/N, as stated in

LEMMA 4.5: *Let F be a finite group acting on a vector space V over a field that is uniquely divisible by the order of F. If W is an F-invariant subspace of V, then*

$$V^F/_{W^F} \cong (V/_W)^F.$$

The proof of this lemma, which consists of an easy cohomological argument, can be found in [7].

Example 4.1 (Example 3.1 continued): Consider any extension (e.g. the semi-direct product)

$$1 \longrightarrow F_{2,4} \longrightarrow E \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow 1$$

of the free 4-generated 2-step nilpotent group $F_{2,4}$ by the group

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle f_1, f_2 | \, f_1 f_2 = f_2 f_1, f_1^2 = f_2^2 = 1 \rangle,$$

where the conjugation action inside E is determined by

$$f_1 e_1 = e_2 f_1, \quad f_1 e_2 = e_1 f_1, \quad f_1 e_3 = e_3 f_1, \quad f_1 e_4 = e_4 f_1$$

and

$$f_2 e_1 = e_1 f_2, \quad f_2 e_2 = e_2 f_1, \quad f_2 e_3 = e_4 f_2, \quad f_2 e_4 = e_3 f_2.$$

Define a module structure $\varphi\colon E \to \mathbb{R}^4$ by

$$\varphi(e_1) = \begin{pmatrix} -1 & 2 & 0 & -2 \\ 2 & -1 & 0 & 2 \\ -2 & 2 & 1 & -2 \\ 4 & -4 & 0 & 5 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} 3 & -2 & 0 & 2 \\ -2 & 3 & 0 & -2 \\ 2 & -2 & 1 & 2 \\ -4 & 4 & 0 & -3 \end{pmatrix},$$

$$\varphi(e_3) = \begin{pmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 5 & 2 \\ 0 & 0 & -6 & -2 \end{pmatrix}, \quad \varphi(e_4) = \begin{pmatrix} 5 & 0 & 0 & 2 \\ -2 & 1 & 0 & -1 \\ 4 & 0 & 1 & 2 \\ -6 & 0 & 0 & -2 \end{pmatrix}.$$

$$\varphi(f_1) = \begin{pmatrix} 5 & 2 & 0 & 2 \\ -4 & -1 & 0 & -2 \\ 4 & 2 & 1 & 2 \\ -8 & -4 & 0 & -3 \end{pmatrix}, \quad \varphi(f_2) = \begin{pmatrix} -3 & 0 & -4 & -4 \\ 2 & 1 & 2 & 2 \\ -4 & 0 & -3 & -4 \\ 6 & 0 & 6 & 7 \end{pmatrix}.$$

Using the computations from Example 3.1 we work out the cohomology of E with coefficients in $\mathbb{R}^4$. The dimensions of the cohomology spaces are given by

$$\begin{array}{lll} H^0(E,\mathbb{R}^4)\cong 0, & H^1(E,\mathbb{R}^4)\cong 0, & H^2(E,\mathbb{R}^4)\cong \mathbb{R}^4, \\ H^3(E,\mathbb{R}^4)\cong \mathbb{R}^{10}, & H^4(E,\mathbb{R}^4)\cong \mathbb{R}^{16}, & H^5(E,\mathbb{R}^4)\cong \mathbb{R}^{19}, \\ H^6(E,\mathbb{R}^4)\cong \mathbb{R}^{21}, & H^7(E,\mathbb{R}^4)\cong \mathbb{R}^{16}, & H^8(E,\mathbb{R}^4)\cong \mathbb{R}^6, \\ H^9(E,\mathbb{R}^4)\cong \mathbb{R}^2, & H^{10}(E,\mathbb{R}^4)\cong \mathbb{R}. & \end{array}$$

5. Cohomology of virtually abelian groups

Let E be a finite extension of the free abelian group $\mathbb{Z}^n$ on n generators. In this section we give an explicit formula for the dimension of the cohomology spaces of E with coefficients in a finite-dimensional real vector space $\mathbb{R}^k$ carrying a particular type of E-module structure. This formula generalises the formula for the Betti numbers of a virtually abelian group as developed in [7] to more general coefficient modules.

Let us first of all describe a polynomial crystallographic action of E on $\mathbb{R}^n$. The extension

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow E \longrightarrow F = E/\mathbb{Z}^n \longrightarrow 0$$

determines an action $\psi\colon F \to \mathrm{GL}(k,\mathbb{Z})$ of F on $\mathbb{Z}^n$ by conjugation inside E. Let $m\colon F\times F \to \mathbb{Z}^n$ be the cocycle corresponding to the extension. Considered as a mapping from $F\times F$ to $\mathbb{R}^n$, m is a cocycle as well. But $H^2(F,\mathbb{R}^n)$ is trivial for F is finite and $\mathbb{R}^n$ is uniquely divisible by the index of F, so m is actually a coboundary. Let $\lambda\colon F\to\mathbb{R}^n$ be a mapping such that $\delta\lambda = m$, that is,

$$\delta\lambda(f,f') = \lambda(f) + \psi(f)\lambda(f') - \lambda(ff') = m(f,f')$$

for all $f, f' \in F$.

THEOREM 5.1 (with notation as introduced above): *The mapping* $\rho\colon E = \mathbb{Z}^n \times_m F \to \mathrm{Aff}(n,\mathbb{R})$ *given by*

$$\rho(z,f) = \left(\begin{array}{ccc|c} & \psi(f) & & z+\lambda(f) \\ \hline 0 & \dots & 0 & 1 \end{array}\right)$$

is an affine crystallographic action.

For a proof of this theorem we refer to [7].

We need some extra concepts and definitions to state a formula for the dimension of $H^*(E,\mathbb{R}^k)$. Let $n, p\in\mathbb{N}$ and $p\le n\ne 0$. Then $\mathcal{S}(p,n-p)$ is the set of all

$(p,n-p)$-shuffles, where a $(p,n-p)$-shuffle is a permutation σ of $\{1,2,\ldots,n\}$ with

$$\sigma(1) < \sigma(2) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \sigma(p+2) < \cdots < \sigma(n).$$

Now let R be any commutative ring and $A = (a_{i,j}) \in R^{n\times n}$ a square n-dimensional matrix over R. Then, for every couple of $(p,n-p)$-shuffles (σ,τ), we set

$$A_{(\sigma,\tau)} = \det \begin{pmatrix} a_{\sigma(1)\tau(1)} & a_{\sigma(1)\tau(2)} & \cdots & a_{\sigma(1)\tau(p)} \\ a_{\sigma(2)\tau(1)} & a_{\sigma(2)\tau(2)} & \cdots & a_{\sigma(2)\tau(p)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\sigma(p)\tau(1)} & a_{\sigma(p)\tau(2)} & \cdots & a_{\sigma(p)\tau(p)} \end{pmatrix}.$$

Listing all $(p,n-p)$-shuffles as

$$\sigma_1, \sigma_2, \ldots, \sigma_c \quad \text{with } c = \binom{n}{p},$$

we define

$$A^{(p)} = \begin{pmatrix} A_{(\sigma_1,\sigma_1)} & A_{(\sigma_1,\sigma_2)} & \cdots & A_{(\sigma_1,\sigma_c)} \\ A_{(\sigma_2,\sigma_1)} & A_{(\sigma_2,\sigma_2)} & \cdots & A_{(\sigma_2,\sigma_c)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{(\sigma_c,\sigma_1)} & A_{(\sigma_c,\sigma_2)} & \cdots & A_{(\sigma_c,\sigma_c)} \end{pmatrix}.$$

PROPOSITION 5.2: *Let E be a finitely generated virtually abelian group with a rank n abelian normal subgroup of finite index. Suppose the conjugation action of $F = E/\mathbb{Z}^n$ on $\mathbb{Z}^n$ is given by ψ: $F \to \mathrm{GL}(n,\mathbb{Z})$, and the module structure φ: $E \to \mathrm{GL}(k,\mathbb{R})$ is such that $\varphi(\mathbb{Z}^n) = \{1\}$. Let F be generated by $a_1, a_2, \ldots, a_r$. Then the dimension of $H^p(E,\mathbb{R}^k)$ is given by*

$$k\binom{n}{p} - \mathrm{Rank}\left[\begin{pmatrix} \varphi(a_1)_{1,1}\psi(a_1^{-1})^{(p)} & \ldots & \varphi(a_1)_{1,k}\psi(a_1^{-1})^{(p)} \\ \vdots & \ddots & \vdots \\ \varphi(a_1)_{k,1}\psi(a_1^{-1})^{(p)} & \ldots & \varphi(a_1)_{k,k}\psi(a_1^{-1})^{(p)} \end{pmatrix} - \mathbb{I}, \ldots,\right.$$
$$\left.\begin{pmatrix} \varphi(a_r)_{1,1}\psi(a_r^{-1})^{(p)} & \ldots & \varphi(a_r)_{1,k}\psi(a_r^{-1})^{(p)} \\ \vdots & \ddots & \vdots \\ \varphi(a_r)_{k,1}\psi(a_r^{-1})^{(p)} & \ldots & \varphi(a_r)_{k,k}\psi(a_r^{-1})^{(p)} \end{pmatrix} - \mathbb{I}\right].$$

The matrix of which we take the rank in this proposition is obtained by juxtaposition of the matrices in the formula; the matrix $\mathbb{I}$ is the identity matrix of appropriate dimension. The proof of this proposition uses the lemma below. Proof of the lemma is left to the reader.

LEMMA 5.3 (with notation as above): *Let $\sigma \in \mathcal{S}(p, n-p)$ and write dx_σ for $dx_{\sigma(1)} \wedge \cdots \wedge dx_{\sigma(p)}$. Then*

$$\rho(e)^*(dx_\sigma) = \sum_{\tau \in \mathcal{S}(p,n-p)} [\rho(e^{-1})_*]_{(\sigma,\tau)} dx_\tau$$

for any $e \in E$. Consequently, for any differential p-form

$$\omega = \sum_{\sigma \in \mathcal{S}(p,n-p)} \omega_\sigma dx_\sigma$$

on $\mathbb{R}^n$ and any $e \in E$ we have

$$\rho(e)^*\omega = \sum_{\tau \in \mathcal{S}(p,n-p)} dx_\tau \Bigg[\sum_{\sigma \in \mathcal{S}(p,n-p)} (\omega_\sigma \circ \rho(e))[\rho(e)_*]_{(\sigma,\tau)} \Bigg].$$

Proof of Proposition 5.2: Let us take the affine crystallographic action $\rho: E \to \mathrm{Aff}(n, \mathbb{R})$ from Theorem 5.1. Then ρ maps $\mathbb{Z}^n$ to translations of $\mathbb{R}^n$. As φ is trivial on $\mathbb{Z}^n$, any $\mathbb{Z}^n$-invariant k-tuple of polynomial p-forms $\omega = {}^t(\omega_1 \ldots \omega_k)$ consists of k constant p-forms

$$\omega_i = \sum_{\sigma \in \mathcal{S}(p,n-p)} c_\sigma^{(i)} dx_\sigma$$

on $\mathbb{R}^n$.

We now determine the F-invariant k-tuples of forms. Clearly, an ω that is invariant under the action of a set of generators $\{a_1, a_2, \ldots, a_r\}$ for F is invariant under the action of any element of F. Let us have a look at the action of an a_i. Choose an $\tilde{a}_i \in E$ such that $\tilde{a}_i N = a_i$. Then

$$({8}) \qquad {}^{a_i}\omega = \varphi(\tilde{a}_i) \cdot \begin{pmatrix} \rho(\tilde{a}_i^{-1})^*\omega_1 \\ \vdots \\ \rho(\tilde{a}_i^{-1})^*\omega_k \end{pmatrix}$$

and, for any $j \in \{1, \ldots, k\}$, we know from Lemma 5.3 that

$$\begin{aligned} \rho(\tilde{a}_i^{-1})^*\omega_i &= \sum_{\tau \in \mathcal{S}(p,n-p)} dx_\tau \Bigg[\sum_{\sigma \in \mathcal{S}(p,n-p)} c_\sigma^{(i)} \psi(a_i^{-1})_{(\sigma,\tau)} \Bigg] \\ &= \begin{pmatrix} dx_{\tau_1} & \ldots & dx_{\tau_{\binom{n}{p}}} \end{pmatrix} \cdot \psi(a_i^{-1})^{(p)} \cdot \begin{pmatrix} c_{\tau_1} \\ \vdots \\ c_{\tau_{\binom{n}{p}}} \end{pmatrix}. \end{aligned}$$

Inserting this result in (8) and subtracting ω again yields the desired result. ∎

References

[1] Y. Benoist and K. Dekimpe, *The uniqueness of polynomial crystallographic actions,* Mathematische Annalen **322** (2002), 563–571.

[2] K. S. Brown, *Cohomology of Groups,* Volume 87 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1982.

[3] K. Dekimpe, *Any virtually polycyclic group admits a NIL-affine crystallographic action,* Topology **42** (2003), 821–832.

[4] K. Dekimpe and P. Igodt, *Polynomial structures for nilpotent groups,* Transactions of the American Mathematical Society **348** (1996), 77–97.

[5] K. Dekimpe and P. Igodt, *Polycyclic-by-finite groups admit a bounded-degree polynomial structure,* Transactions of the American Mathematical Society **349** (1997), 3597–3610.

[6] K. Dekimpe, P. Igodt and K. B. Lee, *Polycyclic-by-finite groups admit a bounded-degree polynomial structure,* Inventiones Mathematicae **129** (1997), 121–140.

[7] K. Dekimpe and H. Pouseele, *The real cohomology of virtually nilpotent groups,* Preprint, 2002.

[8] G. de Rham, *Sur l'analysis situs des variétés à n dimensions,* Journal de Mathématiques Pures et Appliquées **10** (1931), 115–200.

[9] D. Fried, W. Goldman and M. Hirsch, *Affine manifolds with nilpotent holonomy,* Commentarii Mathematici Helvetici **56** (1981), 487–523.

[10] S. Mac Lane, *Homology,* Volume 114 of Die Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1975.

[11] W. S. Massey, *Singular Homology Theory,* Volume 70 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1980.

[12] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups,* Annals of Mathematics **59** (1954), 531–538.

[13] D. Segal, *Polycyclic Groups,* Cambridge University Press, 1983.

[14] G. Whitehead, *Elements of Homotopy Theory,* Volume 61 of Graduate Texts in Mathematics, Springer-Verlag, Berlin, 1978.